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A CONSENSUS MODEL FOR GROUP DECISION MAKING PROBLEMS WITH UNBALANCED FUZZY LINGUISTIC INFORMATION

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Most group decision making (GDM) problems based on linguistic approaches use symmetrically and uniformly distributed linguistic term sets to express experts' opinions. However, there exist problems whose assessments need to be represented by means of unbalanced linguistic term sets, i.e. using term sets that are not uniformly and symmetrically distributed. The aim of this paper is to present a consensus model for GDM problems with unbalanced fuzzy linguistic information. This consensus model is based on both a fuzzy linguistic methodology to deal with unbalanced linguistic term sets and two consensus criteria, consensus degrees, and proximity measures. To do so, we use a new fuzzy linguistic methodology improving another approach to manage unbalanced fuzzy linguistic information,[1] (*Int. J. Intell. Syst.* **22**(11) (2007) 1197–1214), which uses the linguistic 2-tuple model as representation base of unbalanced fuzzy linguistic information. In addition, the consensus model presents a feedback mechanism to help experts for reaching the highest degree of consensus possible. There are two main advantages provided by this consensus model. First, its ability to cope with GDM problems with unbalanced fuzzy linguistic information overcoming the problem of finding different discrimination levels in linguistic term sets. Second, it supports the consensus process automatically, avoiding the possible subjectivity that the moderator can introduce in this phase.

Keywords: Fuzzy linguistic modeling; unbalanced linguistic term set; group decision making; consensus.

1. Introduction

It is undeniably true that group decision making (GDM) problems arise from many real-world situations.[2] In these problems, there are a set of alternatives to solve a problem and a group of experts characterized by their background and knowledge trying to achieve a common solution. To solve these problems, the experts are faced by applying two processes before obtaining a final solution[3–8]: *the consensus process* and *the selection process* (see Fig. 1). The former consists in obtaining the maximum degree of consensus or agreement between the set of experts on the solution set of alternatives. Normally, the consensus process is guided by a human figure called moderator[3,6] who is a person that does not participate in the discussion but monitors the agreement in each moment of the consensus process and is in charge of supervising and addressing the consensus process toward success, i.e. to achieve the maximum possible agreement and to reduce the number of experts outside of the consensus in each new consensus round. The latter refers to obtaining the solution set of alternatives from the opinions on the alternatives given by the experts. Clearly, it is preferable that the set of experts achieves a great agreement among their opinions before applying the selection process, and therefore, in this paper, we focus on the consensus process.

A consensus process is defined as a dynamic and iterative group discussion process, coordinated by a moderator helping experts bring their opinions closer. If the consensus level is lower than a specified threshold, the moderator would urge experts to discuss their opinions further in an effort to bring them closer. On the contrary, when the consensus level is higher than the threshold, the moderator
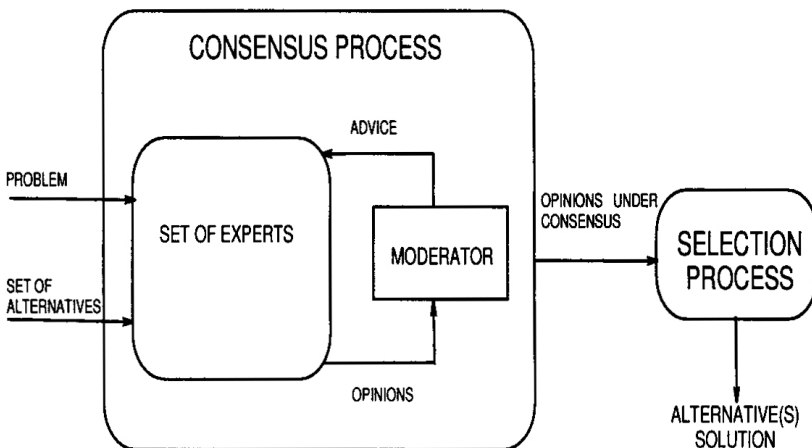
Fig. 1. Resolution process of a GDM problem.

would apply the selection process in order to obtain the final consensus solution to the GDM problem. In this framework, an important question is how to substitute the actions of the moderator in the group discussion process in order to automatically model the whole consensus process. Some approaches have been proposed in GDM with multiple representation formats,[4] in GDM with multi-granular linguistic information,[9] and in GDM with incomplete information.[5, 10]

On the other hand, usually many problems present quantitative aspects, which can be assessed by means of precise numerical values.[2, 7] However, some problems present also qualitative aspects that are complex to assess by means of precise and exact values. In these cases, the fuzzy linguistic approach[9, 11–20] can be used to obtain a better solution. This is the case, for instance, when experts try to evaluate the "comfort" of a car, where linguistic terms like "good," "fair," "poor" are used.[21] Many of these problems use linguistic variables assessed in linguistic term sets whose terms are uniformly and symmetrically distributed, i.e. assuming the same discrimination levels on both sides of mid linguistic term. However, there exist problems that need to assess their variables with linguistic term sets that are not uniformly and symmetrically distributed.[1, 22] This type of linguistic term sets are called *unbalanced linguistic term sets* (see Fig. 2).

The aim of this paper is to present a consensus model for GDM problems where the experts provide their opinions by means of unbalanced fuzzy linguistic information. In Ref. 1, a methodology to manage unbalanced fuzzy linguistic information in information retrieval systems was presented, which used hierarchical linguistic contexts based on the linguistic 2-tuple computational model.[23, 24] However, this methodology can only represent unbalanced linguistic term sets when there exists a level with an adequate granularity to represent the subset of linguistic terms on the left of the mid linguistic term and a level with an adequate granularity to represent the subset of linguistic terms on the right of the mid linguistic term. Thus, we present a new fuzzy linguistic methodology that can represent unbalanced fuzzy linguistic information when the above conditions are not satisfied. As a part of the consensus model, a feedback mechanism substituting the figure of the moderator is given to help experts change their opinions on the alternatives in order to obtain the highest degree of consensus possible. The feedback mechanism consists of simple and easy rules that generate recommendations in the discussion process. Moreover, this model is based on two types of consensus criteria, *consensus degrees* evaluating the agreement of all the experts and deciding when the consensus process should stop, and *proximity measures* evaluating the distance between experts' individual opinions and the group or collective opinion, which is also used in the feedback mechanism to guide the direction of the changes in experts' opinions in
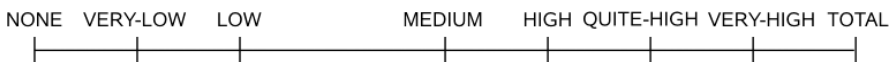

Fig. 2. Example of an unbalanced linguistic term set of eight labels.

order to increase the consensus degrees. Both measures are computed at the three different levels of representation of information of a preference relation: pair of alternatives, alternative, and relation.

In order to do this, the paper is structured as follows. In Sec. 2, we present the new fuzzy linguistic methodology to manage unbalanced fuzzy linguistic information. Section 3 introduces the new consensus model for GDM problems with unbalanced fuzzy linguistic information. In Sec. 4, a practical example is given to illustrate the application of the consensus model. Finally, some concluding remarks are pointed out in Sec. 5.

2. A New Fuzzy Linguistic Methodology to Manage Unbalanced Fuzzy Linguistic Information

In this section, we make a review of the 2-tuple fuzzy linguistic representation model[23] and the concept of hierarchical linguistic contexts[24] in order to present the new methodology to manage unbalanced fuzzy linguistic information.

2.1. *The 2-tuple fuzzy linguistic representation model*

The 2-tuple fuzzy linguistic representation model was introduced in Ref. 23 to carry out processes of computing with words (CW) in a precise way when the linguistic term sets are symmetrically and uniformly distributed and to improve several aspects of the ordinal fuzzy linguistic approach.[25–28] This model is based on the concept of *symbolic translation* and represents the linguistic information by means of a pair of values, (s, α), where s is a linguistic label and α is a numerical value that represents the value of the symbolic translation.

Definition 2.1 (Herrera and Martínez, Ref. 23).   Let $\beta \in [0, g]$ be the result of an aggregation of the indexes of a set of labels assessed in a linguistic term set $S = \{s_0, s_1, \ldots, s_{g-1}, s_g\}$, where g stands for cardinality of S, i.e. the result of a symbolic aggregation operation. Let $i = \mathrm{round}(\beta)$ and $\alpha = \beta - i$ be two values, such that, $i \in [0, g]$ and $\alpha \in [-0.5, 0.5)$, then α is called a symbolic translation.

This model defines a set of transformation functions to manage the linguistic information expressed by linguistic 2-tuples.

Definition 2.2.   Let S be a linguistic term set and $\beta \in [0, g]$ a value supporting the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to β is obtained with the following function:

$$
\begin{aligned}
\Delta &: [0, g] \to S \times [-0.5, 0.5), \\
\Delta(\beta) &= (s_i, \alpha), \\
i &= \mathrm{round}(\beta), \\
\alpha &= \beta - i,
\end{aligned}
\tag{1}
$$

where "round" is the usual round operation, s_i has the closest index label to "β," and "α" is the value of the symbolic translation.

Proposition 2.1. *Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set and (s_i, α) be a linguistic 2-tuple. There is always a function Δ^{-1}, such that, from a 2-tuple value it returns its equivalent numerical value $\beta \in [0, g] \subset \mathbb{R}$:*

$$\Delta^{-1} : S \times [-0.5, 0.5) \to [0, g], \tag{2}$$
$$\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta.$$

Remark 2.1. We should point out that a linguistic term can be seen as a linguistic 2-tuple by adding to it the value 0 as symbolic translation, $s_i \in S \Rightarrow (s_i, 0)$.

The 2-tuples linguistic computational model presents different techniques to manage the linguistic information[23]:

(i) *A 2-tuple comparison operator*: The comparison of linguistic information represented by 2-tuples is carried out according to an ordinary lexicographic order. Let (s_k, α_1) and (s_l, α_2) be two 2-tuples, with each one representing a counting of information:

  (a) if $k < l$ then (s_k, α_1) is smaller than (s_l, α_2).
  (b) if $k = l$ then

    (i) if $\alpha_1 = \alpha_2$ then (s_k, α_1), (s_l, α_2) represent the same information.
    (ii) if $\alpha_1 < \alpha_2$ then (s_k, α_1) is smaller than (s_l, α_2).
    (iii) if $\alpha_1 > \alpha_2$ then (s_k, α_1) is bigger than (s_l, α_2).

- *A 2-tuple negation operator*: It is defined as

$$\text{Neg}(s_i, \alpha) = \Delta(g - \Delta^{-1}(s_i, \alpha)). \tag{3}$$

- *2-tuple aggregation operators*: Using the function Δ and Δ^{-1}, any aggregation operator can be easily extended for dealing with linguistic 2-tuples, such as the LOWA operator,[25] the weighted average operator, the OWA operator, etc. (see Ref. 23).

2.2. *Hierarchical linguistic contexts*

In Ref. 24, the hierarchical linguistic contexts were introduced to improve the precision of processes of CW in multi-granular linguistic contexts.[29,30] In this work, we use them to manage the unbalanced fuzzy linguistic information.

A *linguistic hierarchy* is a set of levels, where each level represents a linguistic term set with different granularity from the remaining levels of the hierarchy. Each level is denoted as $l(t, n(t))$, where t is a number indicating the level of the hierarchy, and $n(t)$ is the cardinality of the linguistic term set of t. Moreover, we assume levels containing linguistic terms whose membership functions are triangular-shaped, uniformly and symmetrically distributed in $[0, 1]$, and linguistic term sets having an
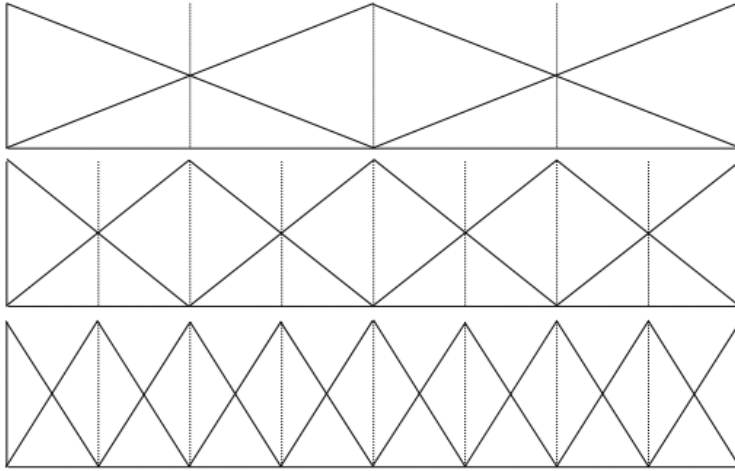
Fig. 3. Linguistic hierarchy of 3, 5, and 9 labels.

odd value of granularity where the central label represents the value of *indifference*. A graphical example of a linguistic hierarchy is shown in Fig. 3.

The levels belonging to a linguistic hierarchy are ordered according to their granularity, i.e. for two consecutive levels t and $t+1$, $n(t+1) > n(t)$. Hence, the level $t+1$ could be considered as a refinement of the previous level t. Then, a linguistic hierarchy LH can be defined as the union of all levels t:

$$LH = \bigcup_t l(t, n(t)). \tag{4}$$

Given an LH, we denote as $\mathcal{S}^{n(t)}$ the linguistic term set of LH corresponding to the level t of LH characterized by a granularity of uncertainty $n(t)$: $\mathcal{S}^{n(t)} = \{s_0^{n(t)}, \ldots, s_{n(t)-1}^{n(t)}\}$. Furthermore, the linguistic term set of the level $t+1$ is obtained from its predecessor as

$$l(t, n(t)) \rightarrow l(t+1, 2 \cdot n(t) - 1). \tag{5}$$

Transformation functions between labels from different levels to make processes of CW in multigranular linguistic information contexts without loss of information were defined in Ref. 24.

Definition 2.3 (Herrera and Martínez, Ref. 23). Let $LH = \bigcup_t l(t, n(t))$ be a linguistic hierarchy whose linguistic term sets are denoted as $\mathcal{S}^{n(t)} = \{s_0^{n(t)}, \ldots, s_{n(t)-1}^{n(t)}\}$, and let us consider the 2-tuple fuzzy linguistic representation. The transformation function from a linguistic label in level t to a label in level t' is defined as $TF_{t'}^t : l(t, n(t)) \rightarrow l(t', n(t'))$ such that

$$TF_{t'}^t(s_i^{n(t)}, \alpha^{n(t)}) = \Delta_{t'}\left(\frac{\Delta_t^{-1}(s_i^{n(t)}, \alpha^{n(t)}) \cdot (n(t') - 1)}{n(t) - 1}\right). \tag{6}$$

2.3. *A new model to manage unbalanced fuzzy linguistic information*

A new model to manage unbalanced fuzzy linguistic term sets based on the linguistic 2-tuple model is presented. It carries out computational operations of unbalanced fuzzy linguistic information using the 2-tuple computational model and different levels of an LH. This new model presents two components:

- a *representation model of unbalanced fuzzy linguistic information* and
- a *computational model of unbalanced fuzzy linguistic information*.

As aforementioned, this methodology overcomes the problem of that model proposed in Ref. 1 because it represents unbalanced fuzzy linguistic information when there does not exist a level in an LH with an adequate granularity to represent the subset of linguistic terms on the left of the mid linguistic term or on the right of the mid linguistic term.

2.3.1. *An unbalanced fuzzy linguistic representation model*

The procedure to represent unbalanced fuzzy linguistic information defined in Ref. 1 works as follows:

(i) Find a level t^- of LH to represent the subset of linguistic terms $\mathcal{S}_{un}^L$ on the left of the mid linguistic term of unbalanced fuzzy linguistic term set $\mathcal{S}_{un}$. This level of LH should support the distribution of the labels of $\mathcal{S}_{un}^L$ on the discourse universe.

(ii) Find a level t^+ of LH to represent the subset of linguistic terms $\mathcal{S}_{un}^R$ on the right of the mid linguistic term of $\mathcal{S}_{un}$.

(iii) Represent the mid term of $\mathcal{S}_{un}$ using the mid terms of the levels t^- and t^+.

As aforementioned, the problem appears when there does not exist a level t^- or t^+ in LH to represent $\mathcal{S}_{un}^L$ or $\mathcal{S}_{un}^R$, respectively. Then, we propose to overcome this problem by applying the following algorithm, which is defined assuming that there does not exist t^-, as it happens with the unbalanced fuzzy linguistic term set given in Fig. 2:

(i) Represent $\mathcal{S}_{un}^L$:
- (a) Identify the mid term of $\mathcal{S}_{un}^L$, called $\mathcal{S}_{mid}^L$. To do so, we have to observe the distribution of the labels of $\mathcal{S}_{un}^L$ on the discourse universe.
- (b) Find a level t_2^- of the left sets of LH^L to represent the left term subset of $\mathcal{S}_{un}^L$, where LH^L represents the left part of LH.
- (c) Find a level t_2^+ of the right sets of LH^L to represent the right term subset of $\mathcal{S}_{un}^L$.
- (d) Represent the mid term $\mathcal{S}_{mid}^L$ using the levels t_2^- and t_2^+.

(ii) Find a level t^+ of LH to represent the subset of linguistic terms $\mathcal{S}_{un}^R$.

(iii) Represent the mid term of $\mathcal{S}_{un}$ using the levels t^+ and t_2^+.
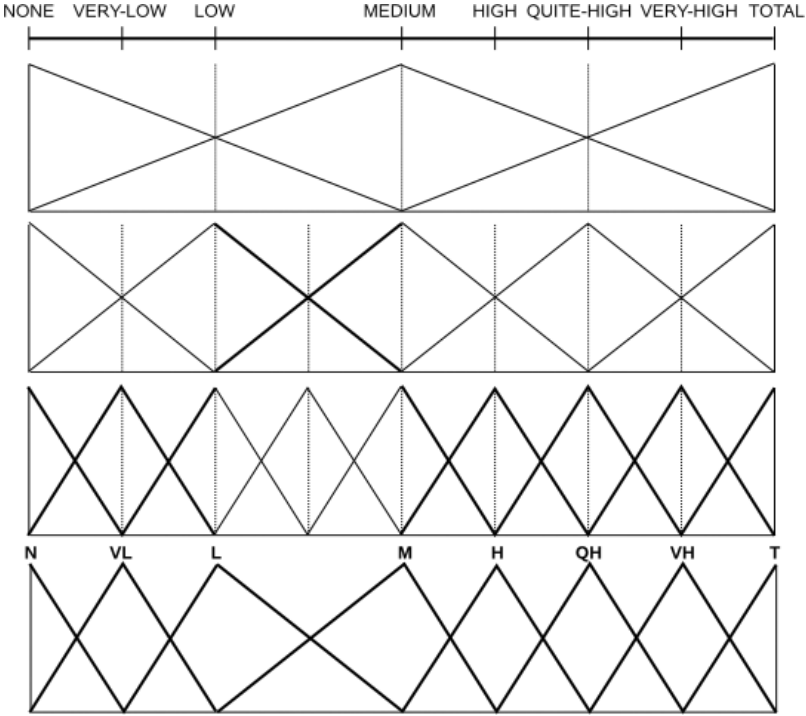
Fig. 4. Representation for an unbalanced term set of eight labels.

For example, applying this algorithm, the representation of the unbalanced fuzzy linguistic term set $\mathcal{S}_{\mathrm{un}} = \{N, VL, L, M, H, QH, VH, T\}$ shown in Fig. 2 with the linguistic hierarchy LH shown in Fig. 3 would be as it is shown in Fig. 4. In this example,

- $\mathcal{S}_{\mathrm{un}}^{L} = \{N, VL, L\}$,
- $\mathcal{S}_{\mathrm{mid}}^{L} = L$,
- $LH^{L} = \{s_0^{n(1)}\} \bigcup \{s_0^{n(2)}, s_1^{n(2)}\} \bigcup \{s_0^{n(3)}, s_1^{n(3)}, s_2^{n(3)}, s_3^{n(3)}\}$.

Thus, we have that $t_2^{-} = 3$, $t_2^{+} = 2$, the mid label $\mathcal{S}_{\mathrm{mid}}^{L} = L$ (due to its position on the discourse universe) is represented using both levels, 3 and 2, and the mid term of $\mathcal{S}_{\mathrm{un}}$ is represented using the levels 2 and 3.

2.3.2. *An unbalanced fuzzy linguistic computational model*

In any fuzzy linguistic approach, we need to define a computational model to manage and aggregate linguistic information. As in Refs. 23, 26, we have to define three types of computation operators to deal with unbalanced fuzzy linguistic information: comparison operators, negation operator, and aggregation operators. In an unbalanced linguistic context, previously to carry out any computation task of unbalanced

fuzzy linguistic information we have to choose a level $t' \in \{t^-, t_2^-, t^+, t_2^+\}$, such that $n(t') = \max\{n(t^-), n(t_2^-), n(t^+), n(t_2^+)\}$:

(i) *An unbalanced linguistic comparison operator*: The comparison of linguistic information represented by two unbalanced linguistic 2-tuples $(s_k^{n(t)}, \alpha_1)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, and $(s_l^{n(t)}, \alpha_2)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$ is similar to the usual comparison of two 2-tuples but acting on the values $TF_{t'}^t(s_k^{n(t)}, \alpha_1) = (s_v^{n(t')}, \beta_1)$ and $TF_{t'}^t(s_l^{n(t)}, \alpha_2) = (s_w^{n(t')}, \beta_2)$. Then, we have

   (a) if $v < w$ then $(s_v^{n(t')}, \beta_1)$ is smaller than $(s_w^{n(t')}, \beta_2)$.
   (b) if $v = w$ then

      (i) if $\beta_1 = \beta_2$ then $(s_v^{n(t')}, \beta_1)$, $(s_w^{n(t')}, \beta_2)$ represent the same information.
      (ii) if $\beta_1 < \beta_2$ then $(s_v^{n(t')}, \beta_1)$ is smaller than $(s_w^{n(t')}, \beta_2)$.
      (iii) if $\beta_1 > \beta_2$ then $(s_v^{n(t')}, \beta_1)$ is bigger than $(s_w^{n(t')}, \beta_2)$.

(ii) *An unbalanced linguistic 2-tuple negation operator*: Let $(s_k^{n(t)}, \alpha)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, be an unbalanced linguistic 2-tuple, then

$$\mathcal{NEG}(s_k^{n(t)}, \alpha) = Neg(TF_{t''}^t(s_k^{n(t)}, \alpha)), \qquad (7)$$

   where $t \neq t'', t'' \in \{t^-, t_2^-, t^+, t_2^+\}$.

(iii) *An unbalanced linguistic aggregation operator*: As aforementioned, in order to deal with unbalanced fuzzy linguistic information we have to represent it in an LH. Hence, any unbalanced linguistic aggregation operator must aggregate unbalanced fuzzy linguistic information by means of its representation in a LH. We use the aggregation processes designed in the 2-tuple computational model but acting on the unbalanced linguistic values transformed by means of $TF_{t'}^t$. Then, once a result is obtained, it is transformed to the correspondent level $t \in \{t^-, t_2^-, t^+, t_2^+\}$ by means of $TF_t^{t'}$ for expressing the result in the unbalanced linguistic term set $\mathcal{S}_{\mathrm{un}}$. In this manner, the LOWA_{un} operator, which is an extension of the linguistic ordered weighted averaging operator,[25] is defined as follows.

Definition 2.4. Let $\{(a_1, \alpha_1), \ldots, (a_m, \alpha_m)\}$ be a set of unbalanced assessments to aggregate, then the $LOWA_{\mathrm{un}}$ operator ϕ_{un} is defined as

$$\phi_{\mathrm{un}}\{(a_1, \alpha_1), \ldots, (a_m, \alpha_m)\} = W \cdot B^{\mathrm{T}} = C_{\mathrm{un}}^m\{w_k, b_k, \ k = 1, \ldots, m\}$$
$$= w_1 \otimes b_1 \oplus (1 - w_1) \otimes C_{\mathrm{un}}^{m-1}\{\beta_h, b_h, \ h = 2, \ldots, m\},$$

where $b_i = (a_i, \alpha_i) \in (\mathcal{S}_{\mathrm{un}} \times [-0.5, 0.5))$, $W = [w_1, \ldots, w_m]$, is a weighting vector, such that $w_i \in [0, 1]$ and $\sum_i w_i = 1, \beta_h = w_h/(\sum_2^m w_k), h = 2, \ldots, m$, and B is the associated ordered unbalanced 2-tuple vector. Each element $b_i \in B$ is the ith largest unbalanced 2-tuple in the collection $\{(a_1, \alpha_1), \ldots, (a_m, \alpha_m)\}$, and C_{un}^m is the convex combination operator of m unbalanced 2-tuples. If $w_j = 1$ and $w_i = 0$ with

$i \neq j \; \forall i, j$ the convex combination is defined as $C_{un}^m \{w_i, b_i, i = 1, \ldots, m\} = b_j$. And if $m = 2$ then it is defined as

$$C_{un}^2 \{w_l, b_l, \; l = 1, 2\} = w_1 \otimes b_j \oplus (1 - w_1) \otimes b_i = TF_t^{t'}(s_k^{n(t')}, \alpha),$$

where $(s_k^{n(t')}, \alpha) = \Delta(\lambda)$ and $\lambda = \Delta^{-1}(TF_{t'}^t(b_i)) + w_1 \cdot (\Delta^{-1}(TF_{t'}^t(b_j)) - \Delta^{-1}(TF_{t'}^t(b_i)))$, $b_j, b_i \in (\mathcal{S}_{un} \times [-0.5, 0.5])$, $(b_j \geq b_i)$, $\lambda \in [0, n(t') - 1]$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$.

In Ref. 31, it was defined an expression to obtain W by means of a fuzzy linguistic nondecreasing quantifier Q^{32}:

$$w_i = Q(i/m) - Q((i-1)/m), \quad i = 1, \ldots, m. \tag{8}$$

3. A Consensus Model for GDM Problems with Unbalanced Fuzzy Linguistic Information

In this section, we present a consensus model defined for GDM problems with unbalanced fuzzy linguistic preference relations providing support to the experts to reach consensus during the process of making a decision. This consensus model presents the following main characteristics:

(i) It is designed to guide the consensus process of unbalanced fuzzy linguistic GDM problems.
(ii) It uses a methodology to manage unbalanced fuzzy linguistic information.
(iii) It is based on two consensus criteria: *consensus degrees* and *proximity measures.* The first ones are used to measure the agreement among all the experts, while the second ones are used to learn how close the collective and individual expert's preference are. Both consensus criteria are calculated at three different levels: pair of alternatives, alternatives, and relation.
(iv) A *feedback mechanism* is defined using the above consensus criteria. It substitutes the moderator's actions, avoiding the possible subjectivity that he/she can introduce and gives advice to the experts to find out the changes they need to make in their opinions in order to obtain the highest degree of consensus possible.

Although the main purpose of our consensus model is to support the experts throughout the consensus process, they are who decide whether or not to follow the advice generated by the consensus model. In any case, in our consensus model the time associated with making the decision is reduced and, therefore, it extends the experts' ability to analyze the information involved in the decision making process. In particular, our consensus model presents three phases that will be described in further detail in the following subsections:

(i) *Computing consensus degrees*: In this phase, the consensus degrees among the experts are calculated. To do so, a similarity measure is defined to obtain the coincidence amongst experts' opinions.

(ii) *Controlling the consensus state*: In this phase, the level of consensus and the number of rounds of discussion to be carried out are checked. In this manner, if the agreement among the experts satisfies a minimum consensus threshold (γ), the consensus process will stop and the selection process will be applied in order to obtain the solution set of alternatives. Otherwise, the third phase is carried out, i.e. the experts' opinions must be modified in order to increase the agreement among the experts. To avoid that the consensus process does not converge after several rounds of discussion, a maximum number of consensus rounds, $MaxRounds$, are incorporated.

(iii) *Feedback mechanism*: To help experts change their opinions, recommendations are provided by means of easy rules. To do this, proximity measures are used in conjunction with the consensus degrees to generate advice so that experts can change their opinions.

3.1. *Computing consensus degrees*

A GDM problem based on preference relations is classically defined as a decision situation where there are a set of experts, $E = \{e_1, \ldots, e_m\}$ ($m \geq 2$), and a finite set of alternatives, $X = \{x_1, \ldots, x_n\}$ ($n \geq 2$), and each expert e_i provides his/her preferences about X by means of a preference relation, $\mathbf{P_{e_i}} \subset X \times X$, where the value $\mu_{\mathbf{P_{e_i}}}(x_j, x_k) = p_i^{jk}$ is interpreted as the preference degree of the alternative x_j over x_k for e_i. In this paper, we deal with unbalanced fuzzy linguistic GDM problems, i.e. GDM problems where the experts e_i express their preferences relations $\mathbf{P_{e_i}} = (p_i^{jk})$ on the set of alternatives X using a linguistic term set that is not uniformly and symmetrically distributed, $\mathcal{S}_{\text{un}}$, which has a minimum label, a maximum label, a central label, and the remaining labels are nonuniformly and nonsymmetrically distributed around the central one, on both left and right lateral sets (as it happens, for example, in the grading system (Fig. 5)). Therefore, $p_i^{jk} \in \mathcal{S}_{\text{un}}$ represents the preference of alternative x_j over alternative x_k for the experts e_i assessed on the unbalanced fuzzy linguistic term set $\mathcal{S}_{\text{un}}$.

Then, consensus degrees are used to measure the current level of consensus in the decision process. As aforementioned, they are given at three different levels: pairs of alternatives, alternatives, and relations. To calculate them, some similarity or coincidence functions are required to obtain the level of agreement among all the experts.[3, 9, 33] Moreover, these similarity functions detect how far each individual expert is from the rest. In this manner, the computation of the consensus degrees is carried out as follows:

(i) For each pair of experts (e_i, e_j) ($i = 1, \ldots, m-1$, $j = i+1, \ldots, m$), a similarity matrix, $SM_{ij} = (sm_{ij}^{lk})$, is defined where

$$sm_{ij}^{lk} = 1 - \frac{|\Delta_{t'}^{-1}(TF_{t'}^{t}(p_i^{lk})) - \Delta_{t'}^{-1}(TF_{t'}^{t}(p_j^{lk}))|}{n(t') - 1}, \tag{9}$$
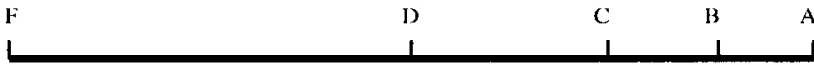
F   D   C   B   A

Fig. 5. Grading system evaluations.

being $p_i^{lk} = (s_v^{n(t)}, \alpha_1)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, $p_j^{lk} = (s_w^{n(t)}, \alpha_2)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, and $t' \in \{t^-, t_2^-, t^+, t_2^+\}$.

(ii) A consensus matrix, $CM = (cm^{lk})$, is calculated by aggregating all the similarity matrices using the arithmetic mean as the aggregation function ϕ:

$$cm^{lk} = \phi(sm_{ij}^{lk}, \ i = 1, \ldots, m-1, \ j = i+1, \ldots, m). \tag{10}$$

(iii) Once the consensus matrix, CM, is computed, we proceed to calculate the consensus degrees at the three different levels:

(a) *Level 1, Consensus degree on pairs of alternatives*: The consensus degree on a pair of alternatives (x_l, x_k), called cp^{lk}, is defined to measure the consensus degree among all the experts on that pair of alternatives. The closer cp^{lk} to 1, the greater the agreement among all the experts on the pair of alternatives (x_l, x_k). Thus, this measure is used to identify those pairs of alternatives with a poor level of consensus and is expressed by the element (l, k) of the consensus matrix CM:

$$cp^{lk} = cm^{lk}, \quad \forall l, k = 1, \ldots, n \wedge l \neq k. \tag{11}$$

(b) *Level 2, Consensus degree on alternatives*: The consensus degree on an alternative x_l, called ca^l, is defined to measure the consensus degree among all the experts on that alternative:

$$ca^l = \frac{\sum_{k=1}^{n} cp^{lk}}{n}. \tag{12}$$

(c) *Level 3, Consensus degree on the relation*: The consensus degree on the relation, called cr, is defined to measure the global consensus degree among all the experts' opinions and is used by the consensus model to control the consensus situation. It is calculated as the average of all the consensus degrees on the alternatives:

$$cr = \frac{\sum_{l=1}^{n} ca^l}{n}. \tag{13}$$

3.2. *Controlling the consensus state*

The consensus state control process involves deciding if the feedback mechanism should be applied to provide advice to the experts or if the consensus process should be finished. To do so, a minimum consensus threshold, $\gamma \in [0, 1]$, is fixed before applying the consensus model. It will depend on the particular problem we are dealing with. On the one hand, when the consequences of the decision to be

made are of a transcendent importance, the minimum level of consensus required to make that decision should be logically as high as possible, and it is not unusual if a minimum value of 0.8 or higher is imposed. On the other hand, if the consequences are not so transcendental (but are still important) or it is urgent to obtain a solution to the problem, a minimum consensus value as close as possible to 0.5 could be required.

In any case, when the consensus measure, cr, satisfies the minimum consensus threshold, γ, the consensus model finishes and the selection process is applied to obtain the solution. Additionally, the consensus model should avoid situations in which the global consensus measure may not converge to the minimum consensus threshold. To do that, a maximum number of rounds *MaxRounds* should be fixed and compared with the current number of round of the consensus model *NumRound*.

In this manner, the operation of the consensus state control process is as follows (see Fig. 6): first, the global consensus measure, cr, is checked against the minimum consensus threshold, γ. If $cr > \gamma$, the consensus process finishes and the selection process is applied. Otherwise, it will check whether the maximum number of rounds, *MaxRounds*, has been reached. If so, it finishes and the selection process is applied too, and if not, it activates the feedback mechanism.
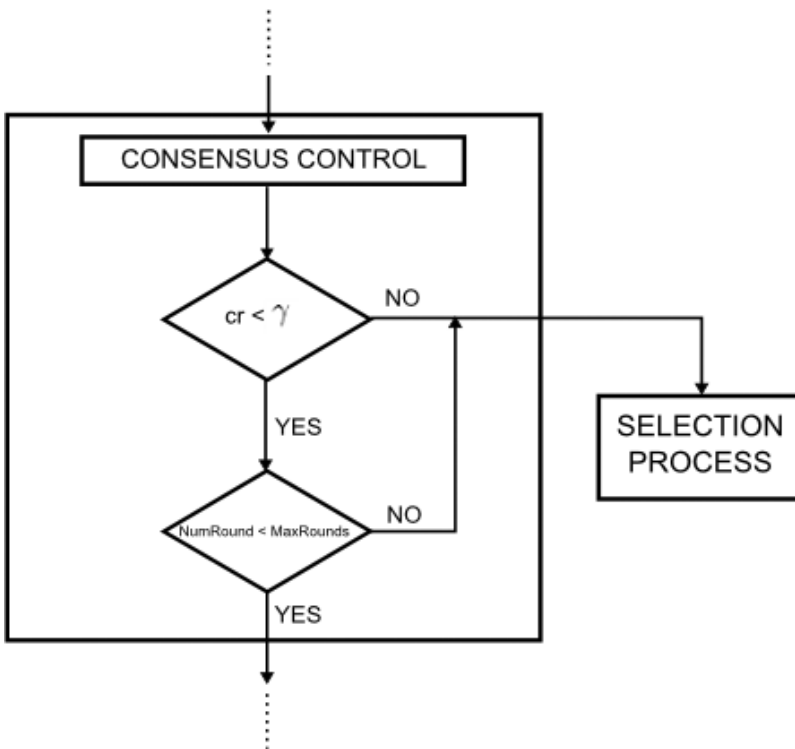

Fig. 6. Consensus control.

3.3.   *Feedback mechanism*

As aforementioned, if the global consensus measure is lower than the minimum consensus threshold, the experts' opinions must be modified. This is done by the feedback mechanism, which provides recommendations to support the experts in changing their opinions. To do so, the feedback mechanism uses proximity measures to identify those experts furthest away from the collective opinion. In the following, both the computation of the proximity measures and the production of advice are explained in detail.

3.3.1. *Computation of proximity measures*

These measures evaluate the agreement between the individual experts' opinions and the group opinion. To compute them for each expert, we need to obtain the collective unbalanced fuzzy linguistic preference relation, $\mathbf{P_{e_c}} = (p_c^{lk})$, which summarizes preferences given by all the experts and is calculated by means of the aggregation of the set of individual unbalanced fuzzy linguistic preference relations $\{\mathbf{P_{e_1}}, \dots, \mathbf{P_{e_m}}\}$ as follows:

$$p_c^{lk} = \phi_{\mathrm{un}}(p_1^{lk}, \dots, p_m^{lk}). \tag{14}$$

with ϕ_{un} the LOWA$_{\mathrm{un}}$ operator defined in Sec. 2.3.

Once $\mathbf{P_{e_c}}$ is obtained, we can compute the proximity measures carrying out the following two steps:

(i) For each expert, e_i, a proximity matrix, $PM_i = (pm_i^{lk})$, is obtained where

$$pm_i^{lk} = 1 - \frac{\left| \Delta_{t'}^{-1}(TF_{t'}^t(p_i^{lk})) - \Delta_{t'}^{-1}(TF_{t'}^t(p_c^{lk})) \right|}{n(t') - 1}, \tag{15}$$

being $p_i^{lk} = (s_v^{n(t)}, \alpha_1), t \in \{t^-, t_2^-, t^+, t_2^+\}, p_c^{lk} = (s_w^{n(t)}, \alpha_2), t \in \{t^-, t_2^-, t^+, t_2^+\}$, and $t' \in \{t^-, t_2^-, t^+, t_2^+\}$.

(ii) Computation of proximity measures at three different levels:

  (a) *Level 1, Proximity measure on pairs of alternatives*: The proximity measure of an expert e_i on a pair of alternatives (x_l, x_k) to the group's one, called pp_i^{lk}, is expressed by the element (l, k) of the proximity matrix PM_i:

$$pp_i^{lk} = pm_i^{lk}, \quad \forall\, l, k = 1, \dots, n \wedge l \neq k. \tag{16}$$

  (b) *Level 2, Proximity measure on alternatives*: The proximity measure of an expert e_i on an alternative x_l to the group's one, called pa_i^l, is calculated as follows:

$$pa_i^l = \frac{\sum_{k=1}^n pp_i^{lk}}{n}. \tag{17}$$

  (c) *Level 3, Proximity measure on the relation*: The proximity measure of an expert e_i on his/her unbalanced fuzzy linguistic preference relation to the

group's one, called pr_i, is calculated as the average of all proximity measures on the alternatives:

$$pr_i = \frac{\sum_{l=1}^{n} pa_i^l}{n}.$$ (18)

The meaning of the proximity measures are as follows: if they are close to 1, then they have a positive contribution for the consensus to be high, while if they are close to 0, then they have a negative contribution to the consensus. Therefore, we can use them to provide advice to the experts to change their opinions and to find out which direction that change has to follow in order to obtain the highest degree of consensus possible.

3.3.2. *Production of advice*

The production of advice to achieve a solution with the highest degree of consensus possible is carried out in two steps: *identification rules* and *direction rules*.

(i) *Identification rules (IR)*: We must identify the experts, alternatives, and pairs of alternatives that are contributing less to reach a high degree of consensus and, therefore, should participate in the change process.

   (a) *Identification rule of experts (IR.1)*: It identifies the set of experts that should receive advice on how to change some of their preference values. This set of experts, called $EXPCH$, that should change their opinions is those whose proximity measure on the relation, pr_i, is lower than the minimum consensus threshold γ. Therefore, the identification rule of experts, IR.1, is as follows:

$$EXPCH = \{i \mid pr_i < \gamma\}.$$ (19)

   (b) *Identification rule of alternatives (IR.2)*: It identifies the alternatives whose associated assessments should be taken into account by the above experts in the change process of their preferences. This set of alternatives is denoted as ALT. The identification rule of alternatives, IR.2, is as follows:

$$ALT = \{x_l \in X \mid ca^l < \gamma\}.$$ (20)

   (c) *Identification rule of pairs of alternatives (IR.3)*: It identifies the particular pairs of alternatives (x_l, x_k) whose respective associated assessments p_i^{lk} the expert e_i should change. This set of pairs of alternatives is denoted as $PALT_i$. The identification rule of pairs of alternatives, IR.3, is as follows:

$$PALT_i = \{(x_l, x_k) \mid x_l \in ALT \wedge e_i \in EXPCH \wedge pp_i^{lk} < \gamma\}.$$ (21)

(ii) *Direction rules (DR)*: We must find out the direction of the change to be recommended in each case, i.e. the direction of change to be applied to the preference assessment p_i^{lk}, with $(x_l, x_k) \in PALT_i$. To do this, we define the following two

direction rules:

(a) *DR.1.* If $p_i^{lk} > p_c^{lk}$, the expert e_i should decrease the assessment associated with the pair of alternatives (x_l, x_k), i.e. p_i^{lk}.

(b) *DR.2.* If $p_i^{lk} < p_c^{lk}$, the expert e_i should increase the assessment associated with the pair of alternatives (x_l, x_k), i.e. p_i^{lk}.

Remark 3.1.   These direction rules will not be produced when a decrease or increase is suggested to an assessment represented by the first or last label of the unbalanced linguistic term set, respectively.

Obviously, the consensus reaching process will depend on the size of the group of experts as well as on the size of the set of alternatives, so that when sizes are small and when opinions are homogeneous, the consensus level required is easier to obtain.[4,34] On the other hand, we note that changes in the experts' opinions will produce a change in the collective opinion, especially when the experts opinions are quite different, i.e. in the early stages of the consensus process. In fact, when experts opinions are close, i.e. when the consensus measure approaches the consensus level required, changes in experts' opinions will not produce a great difference in the collective opinion.

4. Example of Application

For the sake of simplicity, we will assume a low number of experts and alternatives. Let us suppose that four different experts $E = \{e_1, e_2, e_3, e_4\}$ provide the following unbalanced fuzzy linguistic preference relations over a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$ using the unbalanced linguistic term set $\mathcal{S}_{un} = \{N, VL, L, M, H, QH, VH, T\}$ (see Figs. 2 and 4):

$$P_{e_1} = \begin{pmatrix} - & H & QH & L \\ VL & - & M & H \\ L & M & - & L \\ VH & VL & VH & - \end{pmatrix}, \quad P_{e_2} = \begin{pmatrix} - & VL & L & VH \\ H & - & QH & T \\ QH & L & - & H \\ VL & N & L & - \end{pmatrix},$$

$$P_{e_3} = \begin{pmatrix} - & VL & M & VH \\ H & - & QH & L \\ M & L & - & T \\ L & H & N & - \end{pmatrix}, \quad P_{e_4} = \begin{pmatrix} - & VH & QH & M \\ VL & - & M & VH \\ L & M & - & VL \\ M & L & QH & - \end{pmatrix}.$$

Note that the expert did not provide any α values, which is a common practice when expressing preferences with linguistic terms. In these cases, we set $\alpha = 0$:

$$P_{e_1} = \begin{pmatrix} - & (H,0) & (QH,0) & (L,0) \\ (VL,0) & - & (M,0) & (H,0) \\ (L,0) & (M,0) & - & (L,0) \\ (VH,0) & (VL,0) & (VH,0) & - \end{pmatrix},$$

$$P_{e_2} = \begin{pmatrix} - & (VL,0) & (L,0) & (VH,0) \\ (H,0) & - & (QH,0) & (T,0) \\ (QH,0) & (L,0) & - & (H,0) \\ (VL,0) & (N,0) & (L,0) & - \end{pmatrix},$$

$$P_{e_3} = \begin{pmatrix} - & (VL,0) & (M,0) & (VH,0) \\ (H,0) & - & (QH,0) & (L,0) \\ (M,0) & (L,0) & - & (T,0) \\ (L,0) & (H,0) & (N,0) & - \end{pmatrix},$$

$$P_{e_4} = \begin{pmatrix} - & (VH,0) & (QH,0) & (M,0) \\ (VL,0) & - & (M,0) & (VH,0) \\ (L,0) & (M,0) & - & (VL,0) \\ (M,0) & (L,0) & (QH,0) & - \end{pmatrix}.$$

4.1. *First round*

In the following, we show how to apply each step of the consensus model.

(i) *Computing consensus degrees:* First, the similarity matrix for each pair of experts is computed, and so the consensus matrix is obtained. Then, the consensus degrees on pairs of alternatives, alternatives, and global relation are obtained from the consensus matrix.

(a) *Similarity matrices*:

$$SM_{12} = \begin{pmatrix} - & 0.50 & 0.50 & 0.37 \\ 0.50 & - & 0.75 & 0.62 \\ 0.50 & 0.75 & - & 0.62 \\ 0.25 & 0.87 & 0.37 & - \end{pmatrix}, \quad SM_{13} = \begin{pmatrix} - & 0.50 & 0.75 & 0.37 \\ 0.50 & - & 0.75 & 0.62 \\ 0.75 & 0.75 & - & 0.25 \\ 0.37 & 0.50 & 0.12 & - \end{pmatrix},$$

$$SM_{14} = \begin{pmatrix} - & 0.75 & 1.00 & 0.75 \\ 1.00 & - & 1.00 & 0.75 \\ 1.00 & 1.00 & - & 0.87 \\ 0.62 & 0.87 & 0.87 & - \end{pmatrix}, \quad SM_{23} = \begin{pmatrix} - & 1.00 & 0.75 & 1.00 \\ 1.00 & - & 1.00 & 0.25 \\ 0.75 & 1.00 & - & 0.62 \\ 0.87 & 0.37 & 0.75 & - \end{pmatrix},$$

$$SM_{24} = \begin{pmatrix} - & 0.25 & 0.50 & 0.62 \\ 0.50 & - & 0.75 & 0.87 \\ 0.50 & 0.75 & - & 0.50 \\ 0.62 & 0.75 & 0.50 & - \end{pmatrix}, \quad SM_{34} = \begin{pmatrix} - & 0.25 & 0.75 & 0.62 \\ 0.50 & - & 0.75 & 0.37 \\ 0.75 & 0.75 & - & 0.12 \\ 0.75 & 0.62 & 0.75 & - \end{pmatrix}.$$

(b) *Consensus matrix*:

$$CM = \begin{pmatrix} - & 0.54 & 0.70 & 0.62 \\ 0.66 & - & 0.83 & 0.58 \\ 0.70 & 0.83 & - & 0.50 \\ 0.58 & 0.66 & 0.56 & - \end{pmatrix}.$$

(c) *Consensus degrees on pairs of alternatives*: The element (l, k) of CM represents the consensus degrees on the pair of alternatives (x_l, x_k).

(d) *Consensus on alternatives*:

$$ca^1 = 0.62, \quad ca^2 = 0.69, \quad ca^3 = 0.67, \quad ca^4 = 0.60.$$

(e) *Consensus on the relation*:

$$cr = 0.65.$$

(ii) *Controlling the consensus state*: In this step of the consensus model, the global consensus value, cr, is compared with the minimum consensus threshold, γ. In this example, we have decided to use the value, $\gamma = 0.75$. Because $cr < \gamma$, then it is concluded that there is no consensus among the experts, and consequently, the proximity measures are computed in order to support the experts on the necessary changes in their preferences in order to increase cr.

(iii) *Feedback mechanism*: In this step, the proximity measures are computed. To do so, first the collective unbalanced fuzzy linguistic preference relation is obtained by aggregating all individual preference relations. In this case, this is done using the LOW A_{un} operator and the linguistic quantifier *most of* defined as $Q(r) = r^{1/2}$, which applying (8), generates the following weighting vector $\{0.5, 0.20, 0.16, 0.14\}$:

$$P_{e_c} = \begin{pmatrix} - & (M, 0.40) & (H, 0.04) & (QH, -0.36) \\ (M, -0.30) & - & (QH, 0.46) & (QH, -0.46) \\ (M, 0.20) & (M, -0.40) & - & (H, 0.22) \\ (H, -0.38) & (M, -0.49) & (H, -0.26) & - \end{pmatrix}.$$

(a) *Computation of proximity measures*:

(i) *Proximity matrices*:

$$PM_1 = \begin{pmatrix} - & 0.92 & 0.88 & 0.54 \\ 0.66 & - & 0.70 & 0.93 \\ 0.72 & 0.95 & - & 0.60 \\ 0.83 & 0.68 & 0.72 & - \end{pmatrix}, \quad PM_2 = \begin{pmatrix} - & 0.57 & 0.62 & 0.83 \\ 0.83 & - & 0.94 & 0.69 \\ 0.65 & 0.80 & - & 0.97 \\ 0.55 & 0.56 & 0.65 & - \end{pmatrix},$$

$$PM_3 = \begin{pmatrix} - & 0.57 & 0.87 & 0.83 \\ 0.83 & - & 0.94 & 0.56 \\ 0.97 & 0.80 & - & 0.65 \\ 0.67 & 0.81 & 0.41 & - \end{pmatrix}, \quad PM_4 = \begin{pmatrix} - & 0.67 & 0.88 & 0.80 \\ 0.66 & - & 0.70 & 0.82 \\ 0.72 & 0.95 & - & 0.47 \\ 0.92 & 0.81 & 0.84 & - \end{pmatrix}.$$

(ii) *Proximity on pairs of alternatives for expert e_i are given in PM_i.*

(iii) *Proximity on alternatives*: see Table 1.

(iv) *Proximity on the relation*:

$$pr_1 = 0.76, \quad pr_2 = 0.72, \quad pr_3 = 0.74, \quad pr_4 = 0.77.$$

Table 1. Proximity measures on alternatives.

x_1	x_2	x_3	x_4
$pa_1^1 = 0.78$	$pa_1^2 = 0.76$	$pa_1^3 = 0.75$	$pa_1^4 = 0.74$
$pa_2^1 = 0.67$	$pa_2^2 = 0.82$	$pa_2^3 = 0.80$	$pa_2^4 = 0.58$
$pa_3^1 = 0.75$	$pa_3^2 = 0.77$	$pa_3^3 = 0.80$	$pa_3^4 = 0.63$
$pa_4^1 = 0.78$	$pa_4^2 = 0.72$	$pa_4^3 = 0.71$	$pa_4^4 = 0.86$

(b) *Production of advice*:

   (i) *Identification rules*:

     (IR.1) Set of experts to change their preferences, *EXPCH*:
$$EXPCH = \{i \mid pr_i < 0.75\} = \{e_2, e_3\}.$$

     (IR.2) Set of alternatives whose assessments should be considered in the change process, *ALT*:
$$ALT = \{x_l \in X \mid ca^l < 0.75\} = \{x_1, x_2, x_3, x_4\}.$$

     (IR.3) Set of pairs of alternatives whose associated assessments should change, $PALT_i$:

     $PALT_2$
$$= \{(x_1, x_2), (x_1, x_3), (x_2, x_4), (x_3, x_1), (x_4, x_1), (x_4, x_2), (x_4, x_3)\},$$

     which gives the following list of preference values:
$$p_2^{12} \ p_2^{13} \ p_2^{24} \ p_2^{31} \ p_2^{41} \ p_2^{42} \ p_2^{43},$$

     and $PALT_3 = \{(x_1, x_2), (x_2, x_4), (x_3, x_4), (x_4, x_1), (x_4, x_3)\}$, which gives the following list of preference values:
$$p_3^{12} \ p_3^{24} \ p_3^{34} \ p_3^{41} \ p_3^{43}.$$

   (ii) *Direction rules*:
     Because $p_2^{12} < p_c^{12}$, $p_2^{13} < p_c^{13}$, $p_2^{41} < p_c^{41}$, $p_2^{42} < p_c^{42}$, $p_2^{43} < p_c^{43}$, and $p_2^{24} > p_c^{24}$, $p_2^{31} > p_c^{31}$ expert e_2 is advised to increase the assessment of the first five preference values and decrease the assessment of the last two preference values. For the same reason, because $p_3^{12} < p_c^{12}$, $p_3^{24} < p_c^{24}$, $p_3^{41} < p_c^{41}$, $p_3^{43} < p_c^{43}$, and $p_3^{34} > p_c^{34}$ expert e_3 is advised to increase the assessment of the first four preference values and decrease the assessment of the last one preference value.

4.2. *Second round*

   (i) *Providing new preferences*: In this example, we suppose that experts e_2 and e_3 follow the advice given, and thus, their new preferences are as follows:
$$P_{e_2} = \begin{pmatrix} - & (\mathbf{M}, \mathbf{0}) & (\mathbf{H}, \mathbf{0}) & (VH, 0) \\ (H, 0) & - & (QH, 0) & (\mathbf{QH}, \mathbf{0}) \\ (\mathbf{M}, \mathbf{0}) & (L, 0) & - & (H, 0) \\ (\mathbf{M}, \mathbf{0}) & (\mathbf{M}, \mathbf{0}) & (\mathbf{H}, \mathbf{0}) & - \end{pmatrix},$$

$$P_{e_3} = \begin{pmatrix} — & (\mathbf{M}, \mathbf{0}) & (M, 0) & (VH, 0) \\ (H, 0) & — & (QH, 0) & (\mathbf{QH}, \mathbf{0}) \\ (M, 0) & (L, 0) & — & (\mathbf{H}, \mathbf{0}) \\ (\mathbf{M}, \mathbf{0}) & (H, 0) & (\mathbf{H}, \mathbf{0}) & — \end{pmatrix}.$$

(ii) *Computing consensus degrees*:

   (a) *Similarity matrices*:

$$SM_{12} = \begin{pmatrix} — & 0.87 & 0.87 & 0.37 \\ 0.50 & — & 0.75 & 0.87 \\ 0.75 & 0.75 & — & 0.62 \\ 0.62 & 0.62 & 0.75 & — \end{pmatrix}, \quad SM_{13} = \begin{pmatrix} — & 0.87 & 0.75 & 0.37 \\ 0.50 & — & 0.75 & 0.87 \\ 0.75 & 0.75 & — & 0.62 \\ 0.62 & 0.50 & 0.75 & — \end{pmatrix},$$

$$SM_{14} = \begin{pmatrix} — & 0.75 & 1.00 & 0.75 \\ 1.00 & — & 1.00 & 0.75 \\ 1.00 & 1.00 & — & 0.87 \\ 0.62 & 0.87 & 0.87 & — \end{pmatrix}, \quad SM_{23} = \begin{pmatrix} — & 1.00 & 0.87 & 1.00 \\ 1.00 & — & 1.00 & 1.00 \\ 1.00 & 1.00 & — & 1.00 \\ 1.00 & 0.87 & 1.00 & — \end{pmatrix},$$

$$SM_{24} = \begin{pmatrix} — & 0.62 & 0.87 & 0.62 \\ 0.50 & — & 0.75 & 0.87 \\ 0.75 & 0.75 & — & 0.50 \\ 1.00 & 0.75 & 0.87 & — \end{pmatrix}, \quad SM_{34} = \begin{pmatrix} — & 0.62 & 0.75 & 0.62 \\ 0.50 & — & 0.75 & 0.87 \\ 0.75 & 0.75 & — & 0.50 \\ 1.00 & 0.62 & 0.87 & — \end{pmatrix}.$$

   (b) *Consensus matrix*:

$$CM = \begin{pmatrix} — & 0.78 & 0.85 & 0.63 \\ 0.66 & — & 0.83 & 0.87 \\ 0.83 & 0.83 & — & 0.68 \\ 0.81 & 0.71 & 0.85 & — \end{pmatrix}.$$

   (c) *Consensus degrees on pairs of alternatives.* The element (l, k) of CM represents the consensus degrees on the pair of alternatives (x_l, x_k).

   (d) *Consensus on alternatives*:

$$ca^1 = 0.75, \quad ca^2 = 0.79, \quad ca^3 = 0.78, \quad ca^4 = 0.79.$$

   (e) *Consensus on the relation*:

$$cr = 0.78.$$

(iii) *Controlling the consensus state*: As we can observe, the changes in the preference values introduced result in an increasing of the global consensus from 0.65 to 0.78. The minimum consensus threshold is reached, $cr = 0.78 > \gamma = 0.75$, and, therefore, the consensus model would stop and the selection process would be applied to obtain the final solution of consensus.

5. Concluding Remarks

In this paper, we have proposed a consensus model for GDM problems with unbalanced fuzzy linguistic information. There are two main advantages provided by this

consensus model: it is able to manage consensus process in GDM problems with unbalanced fuzzy linguistic information overcoming the problems of unbalanced fuzzy linguistic approach defined in Ref. 1, and it is able to support the consensus process automatically, without moderator, traditionally present in the consensus process.

In the future, we think to extend our consensus model to deal with situations under incomplete information[5, 10] and to build Web-based distributed consensus support systems.[35]

Acknowledgments

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